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Regular Bases in Products of Power Series Spaces

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We prove that $\mathcal{A}_\alpha(\alpha) \times \mathcal{A}_1(\beta)$ has a regular basis iff $\mathcal{A}_\alpha(\alpha) \otimes \mathcal{A}_1(\beta)$ has a regular basis iff $\mathcal{A}_\alpha(\alpha) \otimes \mathcal{A}_1(\beta)$ is isomorphic to a Cartesian product of two power series spaces. We give a simple condition on α, β which determines when these equivalent statements hold.

In 1965 Dragilev [3] introduced the idea of a regular basis in a nuclear Fréchet space. This property, which makes it easy to compute various topological linear invariants, was useful in the study of quasi-equivalence of bases. The culminating result of this development occurs in [2] with the proof that every space with a regular basis has the quasi-equivalence property.

Regular bases have also been interesting in the study of Cartesian products and as a means of classifying nuclear Köthe spaces [1, 4]. In this paper we introduce a property weaker than regularity (we do not know if it is strictly weaker) which avoids certain difficulties occurring in dealing with regularity but is strong enough to obtain all results that can be obtained using regularity. In particular, the result on quasi-equivalence is still valid (Theorem 1).

In his original paper [3], Dragilev gives an example of a Cartesian product of two power series spaces of different type which does not have a regular basis and in [5] he gives an example in which it does. Theorem 2 below gives a complete analysis of the situation for products of power series spaces.

In the last section of this paper we study tensor products of power series spaces and here the concept of pseudo-regularity proves to be a natural tool in characterizing the existence of a regular basis and also in determining when a tensor product can be represented as a Cartesian product, either of two power series spaces or of a space

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of type (d_1) with a space of type (d_2) (Theorem 3). The main value of using pseudo-regularity lies in the fact that the choice of matrix representation is irrelevant. It should be noted that subsequently we have been able to obtain all of our results without using pseudo-regularity.

1. NOTATION AND DEFINITIONS

For the standard theory of nuclear Fréchet spaces and bases in such spaces we refer the reader to [6]. For a lucid description of the diametral dimension $\delta(E)$ and the ideas surrounding the Dragilev theory, see [1].

Two bases (x_n) and (y_n) in a nuclear Fréchet space E are *equivalent* if $\sum t_n x_n$ converges in E iff $\sum t_n y_n$ does. The bases are *semi-equivalent* if there is a sequence of positive numbers (t_n) such that $(t_n x_n)$ is equivalent to (y_n) . They are *quasi-equivalent* if there is a rearrangement of one which is semi-equivalent to the other. A nuclear Fréchet space with a basis has the *quasi-equivalence property* if all bases are quasi-equivalent.

A *representation* of a basis (x_n) in a nuclear Fréchet space E is an infinite matrix (a_n^k) for which there exists a fundamental sequence of seminorms (p_k) defining the topology of E such that $a_n^k = p_k(x_n)$. The basis is *regular* if it has a representation (a_n^k) such that

$$a_{n+1}^k / a_{n+1}^{k+1} \leq a_n^k / a_n^{k+1} \quad \text{for all } k, n,$$

or, equivalently, if there is a representation such that for every k there is an n_k such that the inequality holds for $n \geq n_k$. Such a representation is called a *regular representation*. It is not hard to see that a regular basis can have representations which are not regular.

A basis (x_n) in a nuclear Fréchet space is of *type (d_1)* if it is regular and if, for any representation (a_n^k) , we have

$$\exists p \ni \forall q \ni \exists r \ni \lim_n ((a_n^q)^2 / a_n^r a_n^p) = 0.$$

It is of *type (d_2)* if it is regular and if, for any representation (a_n^k) , we have

$$\forall p \ni \exists q \ni \forall r, \quad \lim_n ((a_n^r)^2 / a_n^q a_n^p) = \infty.$$

The simplest and most important examples of nuclear Fréchet spaces are the power series spaces. Specifically, we will consider

in this paper nondecreasing sequences of positive numbers α, β such that

$$\sup_n \frac{\log(n+1)}{\alpha_n} < \infty \quad \text{and} \quad \lim_n \frac{\log(n+1)}{\beta_n} = 0.$$

Then we define the *power series spaces* $A_\alpha(\alpha)$, $A_1(\beta)$ of *infinite and finite types*, respectively, by

$$A_\alpha(\alpha) = \left\{ \xi = (\xi_n): p_k(\xi) = \sum_n |\xi_n| k^{\alpha_n} < \infty, k = 1, 2, \dots \right\},$$

$$A_1(\beta) = \left\{ \xi = (\xi_n): p_k(\xi) = \sum_n |\xi_n| \left(\frac{k}{k+1} \right)^{\beta_n} < \infty, k = 1, 2, \dots \right\}.$$

With the seminorms (p_k) in each case the spaces are nuclear Fréchet spaces. Moreover, the sequence (e_n) where e_n is the infinite sequence whose n th term is 1 and all others 0, is a basis for each of these spaces, of types (d_1) , (d_2) , respectively. It is called the *coordinate basis*.

In Theorem 3 we will also use sequences $\tilde{\alpha}, \hat{\alpha}$ in the role of α , and sequences $\tilde{\beta}, \hat{\beta}$ in the role of β .

N will refer to the set of positive integers and c_0 will be the set of all those sequences which converge to 0.

If E is a set of sequences and I is a subset of N , then the *stepspace of E corresponding to I* is the set E_I of those $\xi \in E$ such that $\xi_n = 0$ for $n \notin I$.

If λ is a set of sequences we define the Köthe dual λ^\times of λ by

$$\lambda^\times = \left\{ \eta = (\eta_n): \sum_n |\xi_n \eta_n| < \infty \quad \forall \xi \in \lambda \right\}.$$

If a is a sequence and μ is another set of sequences, we write

$$a \cdot \lambda = \{(a_n b_n)_n: b \in \lambda\}, \quad \mu \cdot \lambda = \bigcup_{a \in \mu} a \cdot \lambda.$$

2. PSEUDO-REGULAR BASES

The difficulty which often arises with regularity is that in order to show that a basis is not regular, one must consider all possible representations. To avoid this problem we introduce the following definition. A basis (x_n) in a nuclear Fréchet space is *pseudo-regular* if it has a representation (a_n^k) such that

$$\forall p \exists q \ni \forall r > q \exists s > p \quad \text{and} \quad M > 0 \ni \frac{a_n^p}{a_n^s} < M \left(\frac{a_m^q}{a_m^r} \right) \quad \forall m \leq n.$$

It is easy to check that this holds for a given representation of (x_n) iff it holds for every representation of (x_n) .

For one example in which pseudo-regularity seems more convenient to work with than regularity, one might compare the proof of [3, Theorem 8 (along with Lemma 3)] with the argument for (ii) \Rightarrow (iii) in Theorem 2 below (along with Proposition 5 and the simple fact that condition (iii) of Theorem 2 does not hold for $\alpha_n - \beta_n = n$).

We begin with some results that compare pseudo-regularity with regularity.

PROPOSITION 1. *Every regular basis is pseudo-regular.*

Proof. If (a_n^k) is the representation of the basis which satisfies the definition of regularity, then by taking $q = p$ and $s = r$ we see that the representation also satisfies the definition of pseudo-regularity.

PROBLEM 1. Is every pseudoregular basis regular?

The solution to Problem 1 is positive for a wide class of spaces given by Theorem 3 below. For now we will give one simple illustration.

EXAMPLE 1. If $E = A_\infty(\alpha)$, where $\lim_n (\alpha_n/\alpha_{n+1}) = 0$, then a basis for E is regular iff it is pseudo-regular. Indeed, if π is a permutation of N , then $(e_{\pi(n)})$ is a basis for E and it is pseudo-regular iff

$$\forall p \exists q \ni \forall r > q \exists s > p \ni \left(\frac{p}{s}\right)^{\alpha_{\pi(n)}} < \left(\frac{q}{r}\right)^{\alpha_{\pi(m)}} \quad \forall m \leq n.$$

This, along with our assumption about (α_n) , implies that, with perhaps finitely many exceptions, $\pi(m) \leq \pi(n)$ for $m \leq n$. Hence, $\pi(n) = n$ for n sufficiently large, so that $(e_{\pi(n)})$ is obviously regular. Since E has the quasi-equivalence property, it then follows that every pseudo-regular basis for E is regular.

In view of [2, Lemma 2], there is another property of bases which we might compare with regularity and pseudo-regularity—that is, $\delta(E) = \lambda \cdot \lambda^\times$. We can show that this property is implied by pseudo-regularity.

PROPOSITION 2. *Let E be a nuclear Fréchet space with a pseudo-regular basis (x_n) and a representation (a_n^p) . Let $\lambda = \bigcap_p (1/a^p)l_1$. Then $\delta(E) = \lambda \cdot \lambda^\times$.*

Proof. We may assume that (a_n^p) is such that $a^p/a^q \in c_0$ for $p < q$. Then we can compute directly,

$$\delta(E) = \bigcup_p \bigcap_{s > p} \left(\frac{a^p}{a^s} \right)^\pi \cdot c_0, \quad \lambda \cdot \lambda^\times = \bigcup_q \bigcap_{r > q} \left(\frac{a^q}{a^r} \right) \cdot c_0,$$

where the superscript π refers to the decreasing rearrangement of a sequence in c_0 . Now the definition of pseudo-regularity implies that $\forall p \exists q \exists \forall r > q \exists s > p \ni$ the sequence a^q/a^r dominates the decreasing rearrangement of the sequence a^p/a^s , and the sequence a^p/a^s is dominated by the decreasing rearrangement of a^q/a^r . The first inequality leads us to conclude that $\delta(E) \subset \lambda \cdot \lambda^\times$ and the second that $\lambda \cdot \lambda^\times \subset \delta(E)$, so the proposition is established.

PROBLEM 2. In the context of Proposition 2 does the equality $\delta(E) = \lambda \cdot \lambda^\times$ imply that the basis is regular, or even pseudo-regular?

PROBLEM 3. If a basis (x_n) for a nuclear Fréchet space E has the property that $\delta(E) = \lambda \cdot \lambda^\times$ (λ is defined as in Proposition 2), does it follow that E has the quasi-equivalence property?

We can give an apparently weaker form of Problem 3 which is actually equivalent because of [2, Lemma 2].

PROBLEM 3'. If a basis (x_n) for a nuclear Fréchet space E has the property that $\delta(E) = \lambda \cdot \lambda^\times$, does every basis for E have a permutation which has this property?

3. CARTESIAN PRODUCTS OF POWER SERIES SPACES

In this section we establish some facts which indicate that the theory of regular bases carries over to pseudo-regular bases. These results are used to obtain information about the regularity of bases in product spaces.

PROPOSITION 3. *If a nuclear Fréchet space E has a pseudo-regular basis then every basis for E has a permutation which is pseudo-regular.*

Proof. It is obvious that the inequality in the definition of pseudo-regularity remains true if the matrix (a_n^p) is replaced by $(t_n a_{k_n}^p)$ where (k_n) is any nondecreasing unbounded sequence of integers and (t_n) is any sequence of positive numbers. The result then follows from [1, Theorem 2.2].

PROPOSITION 4. *Any two pseudo-regular bases in a nuclear Fréchet space are semi-equivalent.*

Proof. This is immediate from Proposition 2 and [2, Lemma 2].

COROLLARY. *If a nuclear Fréchet space has a regular basis, then every pseudo-regular basis for this space is regular.*

Proof. Let (x_n) be a regular basis and (y_n) a pseudo-regular basis in a nuclear Fréchet space. From Propositions 1 and 4 it follows that these two bases are semi-equivalent, so (y_n) is regular.

Remark. The corollary extends Example 1 to all power series spaces. We will give below some further cases in which the two properties are equivalent.

THEOREM 1. *Every nuclear Fréchet space with a pseudo-regular basis has the quasi-equivalence property.*

Proof. This is an immediate consequence of Propositions 3 and 4.

Remark. Using the above results along with [1, Theorem 2.2], it is not difficult to extend many of the results of [3, 4], to spaces with pseudo-regular bases. For the remainder of this paper we concern ourselves with Cartesian and tensor products of power series spaces. We begin with a preliminary result that is a generalization of an argument of Dragilev [3].

PROPOSITION 5. *Let E, F be nuclear Fréchet spaces with (alt., pseudo-) regular bases $(x_n), (y_n)$ respectively. Then $E \times F$ has a (pseudo-) regular basis iff there exist strictly increasing sequences of indices (m_i) and (n_i) such that the basis (v_n) given by*

$$\dots, y_{n_i}, x_{m_{i-1}+1}, x_{m_{i-1}+2}, \dots, x_{m_i}, y_{n_{i-1}}, y_{n_{i-2}}, \dots, y_{n_{i+1}}, x_{m_{i+1}}, \dots$$

is (pseudo-) regular. (Note that we are identifying E, F as subspaces of $E \times F$ in the usual way.)

Proof. Suppose that $E \times F$ has a regular basis. By [3, Theorem 1] we know that a regular basis (w_n) can be obtained by joining together the sequences (x_n) and (y_n) in some order. Since a subsequence of a regular basis is regular we obtain regular bases $(x_{\pi(n)}), (y_{\sigma(n)})$ for E, F . That is, (w_n) consists of the terms of the sequences $(x_{\pi(n)}), (y_{\sigma(n)})$ intertwined in some fixed way without changing the order. But then, by [2, Lemma 2], $(x_{\pi(n)})$ is semi-equivalent to (x_n) and

$(y_{o(u)})$ is semi-equivalent to (y_u) . Thus, if we construct (v_u) by putting x_u, y_u where $x_{\pi(u)}, y_{o(u)}$ appeared in (u_u) it follows that (v_u) is semi-equivalent to (u_u) , so it is regular.

The proof for pseudo-regular bases is obtained by using Propositions 3 and 4 instead of [3, Theorem 1; 2, Lemma 2].

THEOREM 2. *The following statements are equivalent.*

- (i) $A_x(\alpha) \times A_1(\beta)$ has a regular basis.
- (ii) $A_x(\alpha) \times A_1(\beta)$ has a pseudo-regular basis.
- (iii) There exist strictly increasing sequences of indices (m_i) and (n_i) such that

$$\lim_i \left(\frac{\alpha_{m_i}}{\beta_{n_i+1}} \right) = 0 \quad \text{and} \quad \inf_i \left(\frac{\alpha_{m_i+1}}{\beta_{n_i+1}} \right) > 0.$$

Proof. (i) \Rightarrow (ii) is just Proposition 1.

(ii) \Rightarrow (iii). We use Proposition 5 with $(x_u), (y_u)$ equal to the coordinate bases for $A_x(\alpha), A_1(\beta)$, respectively. If we then apply the definition of pseudo-regularity to the basis (v_u) with $m = m_i, n = n_i + 1$ we conclude that

$$\rho^{\beta_{n_i+1}} \leq \bar{\rho}^{\alpha_{m_i}} \quad \forall i \text{ sufficiently large and } \rho, \bar{\rho} \in (0, 1),$$

which implies the first part of (iii). For the second part we set $m = n_{i+1}, n = m_i + 1$ and conclude that

$$\exists \rho, \bar{\rho} \in (0, 1) \ni \rho^{\alpha_{m_i+1}} \leq \bar{\rho}^{\beta_{n_{i+1}}} \quad \forall i \text{ sufficiently large,}$$

which implies the second part of (iii).

(iii) \Rightarrow (i). We will show that, using the sequences $(m_i), (n_i)$ from (iii), the basis (v_u) of Proposition 5 is regular. From the second part of (iii) we have $M > 0$ such that $\beta_{n_{i+1}} \leq M\alpha_{m_i+1} \quad \forall i$. Hence we can choose a strictly increasing sequence of positive integers (k_p) such that

$$\frac{k_p}{k_{p-1}} \leq \left(\frac{k_p}{k_p + 1} \right)^M \quad \forall p.$$

Then if we set

$$\begin{aligned} a_u^v &= (k_p)^{\alpha_v} & \text{if } v_u = x_v, \quad m_{i-1} < v \leq m_i, \\ &= \left(\frac{k_p}{k_p + 1} \right)^{\beta_v} & \text{if } v_u = y_v, \quad n_i < v \leq n_{i+1}, \end{aligned}$$

it is not hard to check that (a_u^v) is a representation for (v_u) which satisfies the definition of regularity. This completes the proof of the theorem.

In the next section we will give some specific cases for α, β in which condition (iii) of Theorem 2 holds and some cases in which it does not.

4. TENSOR PRODUCTS OF POWER SERIES SPACES

In this section we characterize those α, β for which $A_\alpha(\alpha) \otimes A_1(\beta)$ has a (pseudo-) regular basis. This will establish the quasi-equivalence property for a large class of tensor products.

If E_1, E_2 are nuclear Fréchet spaces with bases $(x_n), (y_n)$ respectively, then the completed topological tensor product $E_1 \otimes E_2$ is a nuclear Fréchet space with basis $(x_m \otimes y_n)_{(m,n) \in N \times N}$. If $(a_n^k), (b_n^k)$ are matrix representations of $(x_n), (y_n)$, respectively, then $(a_m^k b_n^k)$ is a matrix representation of $(x_m \otimes y_n)$. We will not need any further information about tensor products. For more details we refer the reader to [6].

In this section, (x_n) and (y_n) will refer to the coordinate bases for $A_\alpha(\alpha)$ and $A_1(\beta)$, respectively. We will consider as standard representations of these bases, the matrices $(k^{\alpha_n}), ((k/(k+1))^{\beta_n})$.

PROPOSITION 6. *If $E = A_\alpha(\alpha) \otimes A_1(\beta)$ is isomorphic to a space $F_1 \times F_2$ where F_i is of type (d_i) , $i = 1, 2$, then α, β satisfy condition (iii) of Theorem 2.*

Proof. Using the fact that $F_1 \times F_2$ has the quasi-equivalence property [8], we can show that $N \times N$ is equal to the union of two disjoint subsets I_1 and I_2 such that the stepspace E_{I_i} is isomorphic to F_i , $i = 1, 2$. Thus $A_\alpha(\alpha) \otimes A_1(\beta) = E_{I_1} \oplus E_{I_2}$. Since no stepspace of a space of type (d_i) is of type (d_j) when $i \neq j$ [3], it follows that the sets $\{n: (m, n) \in I_1\}$, $m = 1, 2, \dots$, and $\{m: (m, n) \in I_2\}$, $n = 1, 2, \dots$, are finite. Thus we can define, for each n ,

$$g(n) = \min\{m: (m, n) \in I_1\}, \quad h(n) = \max\{m: (m, n) \in I_2\}.$$

Now let $N_0 = \{n: h(n) > g(n)\}$. Since $(g(n), n) \in I_1$ for all n , we conclude that $\exists p \ni \forall q \exists r \ni$,

$$\begin{aligned} & \sup_{n \in N_0} \left(\frac{q^2}{pr} \right)^{\alpha_{h(n)}} \left[\left(\frac{q}{q+1} \right)^2 \frac{(p+1)(r+1)}{pr} \right]^{\beta_n} \\ & \leq \sup_{n \in N_0} \left(\frac{q^2}{rp} \right)^{\alpha_{g(n)}} \left[\left(\frac{q}{q+1} \right)^2 \frac{(p+1)(r+1)}{pr} \right]^{\beta_n} < \infty. \end{aligned}$$

But $(h(n), n) \in I_2$ for all n so it follows that N_0 is finite. Thus we may assume that $h(n) \leq g(n)$ for all n , which immediately implies that $h(n) = g(n) - 1$.

Next let $N_0 = \{n: g(n+1) < g(n)\}$. It follows that $(h(n), n) \in I_2$ for all n and $(h(n), n+1) \in I_1$ for $n \in N_0$. From the definition of type (d_2) it follows that $\forall p \exists q \ni \forall r$,

$$\inf_{n \in N_0} \left(\frac{q^2}{pr} \right)^{\alpha_{h(n)}} \left[\left(\frac{q}{q+1} \right)^2 \frac{(p+1)(r+1)}{pr} \right]^{\beta_{n+1}} \\ \geq \inf_{n \in N_0} \left(\frac{q^2}{pr} \right)^{\alpha_{h(n)}} \left[\left(\frac{q}{q+1} \right)^2 \frac{(p+1)(r+1)}{pr} \right]^{\beta_n} > 0.$$

Since $(h(n), n+1) \in I_1$ for $n \in N_0$ it follows that N_0 is finite. Thus we may assume that $g(n)$ is nondecreasing.

Now we define the sequences (m_i) , (n_i) inductively by setting $n_1 = \max\{n: h(n) = h(1)\}$, $m_i = h(n_i + 1)$ and $n_{i+1} = \max\{n: h(n) = m_i\}$ for $i = 1, 2, \dots$. This implies that $(m_i + 1, n_{i+1}) \in I_1$ and $(m_i, n_i + 1) \in I_2$ for all i . From the definition of (d_1) and (d_2) we may then conclude that

$$\exists \rho, \bar{\rho} \in (0, 1) \ni \rho^{\alpha_{m_{i+1}}} \leq \bar{\rho}^{\beta_{n_{i+1}}} \text{ for } i \text{ sufficiently large,}$$

and

$$\rho^{\beta_{n_{i+1}}} \leq \bar{\rho}^{\alpha_{m_i}} \text{ for all } i \text{ sufficiently large and } \rho, \bar{\rho} \in (0, 1).$$

This implies the desired conditions and completes the proof of the proposition.

We are now ready for the main result of this paper.

THEOREM 3. *Let $E = A_\infty(\alpha) \otimes A_1(\beta)$. The following conditions are equivalent.*

- (i) E has a regular basis.
- (ii) E has a pseudo-regular basis.
- (iii) $A_\infty(\alpha) \times A_1(\beta)$ has a regular basis.
- (iv) $A_\infty(\alpha) \times A_1(\beta)$ has a pseudo-regular basis.
- (v) *There exist strictly increasing sequences of indices (m_i) and (n_i) such that if $\tilde{\alpha}$ is generated from α by inserting n_i repetitions of α_{m_i} when $m_{i-1} < m \leq m_i$ and $\tilde{\beta}$ is generated from β by inserting m_i repetitions of β_{n_i} when $n_i < n \leq n_{i+1}$, then E is isomorphic to $A_\infty(\tilde{\alpha}) \times A_1(\tilde{\beta})$.*
- (vi) E is isomorphic to a space $F_1 \times F_2$ where F_i is of type (d_i) , $i = 1, 2$.
- (vii) α, β satisfy condition (iii) of Theorem 2.

Proof. The implications (i) \Rightarrow (ii) and (vi) \Rightarrow (vii) are just Propositions 1 and 6, respectively. The equivalence of (iii), (iv), and (vii) is just Theorem 2. The implication (v) \Rightarrow (vi) is obvious and (iv) follows from (ii) because a subsequence of a pseudo-regular basis is obviously pseudo-regular. We will complete the proof by showing that (v) and (iii) imply (i) and (vii) implies (v).

If (iii) holds, then, since $A_\infty(\tilde{\alpha}) \times A_1(\tilde{\beta})$ has a representation which is obtained from a regular representation for $A_\infty(\alpha) \times A_1(\beta)$ by repeating each column a finite number of times, it follows that $A_\infty(\tilde{\alpha}) \times A_1(\tilde{\beta})$ has a regular basis. But then, if (v) holds, E has a regular basis.

Finally then, let us suppose that (vii) holds. Set $m_0 = 0$ and

$$I_1 = \bigcup_{i=1}^{\infty} \{(m, n): m_{i-1} < m \leq m_i, n \leq n_i\},$$

$$I_2 = \bigcup_{i=1}^{\infty} \{(m, n): n_i < m \leq n_{i+1}, m \leq m_i\}.$$

Clearly $N \times N$ is the disjoint union of I_1, I_2 so $E = E_{I_1} \otimes E_{I_2}$. We will show that $E_{I_1} = A_\infty(\hat{\alpha})$, $E_{I_2} = A_1(\hat{\beta})$ where $\hat{\alpha}_{m,n} = \alpha_m$, $(m, n) \in I_1$, and $\hat{\beta}_{m,n} = \beta_n$, $(m, n) \in I_2$. This will complete the proof since $\hat{\alpha}, \hat{\beta}$ are rearrangements of $\tilde{\alpha}, \tilde{\beta}$, respectively.

From (vii) we have $\delta > 0$ such that $\alpha_{m_{i+1}} \geq \delta \beta_{n_{i+1}}$, for all i . Thus, given p we choose $q \geq p$ such that

$$\left(\frac{p}{q}\right)^\delta \leq \frac{q}{q+1},$$

and this implies that

$$\left(\frac{p}{q}\right)^{\alpha_m} \leq \left(\frac{q}{q+1}\right)^{\beta_n}, \quad (m, n) \in I_1,$$

so that $E_{I_1} \subset A_\infty(\hat{\alpha})$. The opposite inclusion is immediate. Next we note that for any p we can choose $\epsilon > 0$ such that

$$p^{1+\epsilon} \leq \frac{(p+1)^2}{p+2},$$

and from (vii) we know that $\alpha_{m_i} \leq \epsilon \beta_{n_{i+1}}$ for i sufficiently large so that

$$k^{\alpha_m} \leq \left(\frac{(k+1)^2}{k(k+2)}\right)^{\beta_n},$$

for all but finitely many $(m, n) \in I_2$. This shows that $A_1(\hat{\beta}) \subset E_{I_2}$ and the opposite inclusion is immediate, so the theorem is proved.

Remarks. (1) We can give two specific cases in which condition (iii) of Theorem 2 is satisfied.

(a) Given α , there exists β such that the condition is satisfied iff $\sup_n(\alpha_{n+1}/\alpha_n) = \infty$.

(b) Given β , there exists α such that the condition is satisfied iff $\sup_n(\beta_{n+1}/\beta_n) = \infty$.

Indeed, the condition implies that the ratios go to ∞ as m, n run through the values m_i, n_i , respectively. For the converse, in case (a) we choose (m_i) such that $\lim_i(\alpha_{m_i+1}/\alpha_{m_i}) = \infty$ and set $n_i = i$, $\beta_i = \alpha_{m_i+1}$. In case (b) we choose (n_i) such that $\lim_i(\beta_{n_i+1}/\beta_{n_i}) = \infty$ and set $m_i = i$, $\alpha_i = \beta_{n_i}$.

(2) Of course, the above conditions do not characterize β in terms of α (or α in terms of β). This is done by observing that condition (iii) of Theorem 2 is equivalent to each of the following statements.

(a) $\exists \delta > 0 \ni \lim_n(1/\beta_n) \max\{\alpha_m : \alpha_m < \delta\beta_n\} = 0$,

(b) $\exists M > 0 \ni \lim_m(1/\alpha_m) \min\{\beta_n : \beta_n > M\alpha_m\} = \infty$.

Indeed, if (a) holds we may take (m_i) and (n_i) as determined by the increasing rearrangement of $\{\alpha_m\} \cup \{2\delta\beta_n\}$ with notation as in Proposition 5. The condition then follows immediately. Conversely, we may consider n such that $n_i < n \leq n_{i+1}$ and suppose that $\alpha_{m_i+1} \geq \delta\beta_{n_i+1}$ for all i . Then if $\alpha_m < \delta\beta_n$ it follows that $\alpha_m \leq \delta\beta_{n_i+1}$, so $m \leq m_i$ and

$$\frac{\alpha_m}{\beta_n} \leq \frac{\alpha_{m_i}}{\beta_{n_{i+1}}} \rightarrow 0.$$

The equivalence of (b) is established similarly.

(3) Let (β_n) be such that $\lim_n(\beta_{n+1}/\beta_n) = \infty$, so that by Remark 1(b) there is an α for which condition (iii) of Theorem 2 holds. However, if we define α by setting $\alpha_n = \beta_n/k$ for $n = 2^{k-1}(2j-1)$, $k, j = 1, 2, \dots$, then, by 2(a), the condition does not hold for this α .

(4) If α, β satisfy condition (iii) of Theorem 2 then $A_x(\alpha) \otimes A_1(\beta)$ satisfies the quasi-equivalence property. This follows from Theorem 3(vii) \Rightarrow (ii) and Theorem 1. Similarly, $A_x(\alpha) \times A_1(\beta)$ also has the quasi-equivalence property. Alternatively, the same result can be obtained by using Zaharjuta's result [8, Theorem 12] and Theorem 3(vii) \Rightarrow (vi).

(5) We conclude by showing that in all cases the space is changed if \otimes is replaced by \times and the exponent sequences are unchanged.

Certain special cases of this fact follow easily from the computations of approximate dimension as, for example, in [7].

PROPOSITION 7. $A_x(\alpha) \otimes A_1(\beta)$ is never isomorphic to $A_x(\alpha) \times A_1(\beta)$.

Proof. If this were so, then in the notation of Theorem 3(v), $A_x(\alpha) \times A_1(\beta)$ would be isomorphic to $A_x(\tilde{\alpha}) \times A_1(\tilde{\beta})$. By a theorem of Zaharjuta [8, Theorem 1], there is an integer s such that $A_x(\alpha)$ is isomorphic to $[A_x(\tilde{\alpha})]^s$, where the superscript s indicates a product with an s -dimensional space or a quotient by a $(-s)$ -dimensional space depending on whether s is positive or negative. This implies that

$$0 < \inf_m \frac{\alpha_m}{\tilde{\alpha}_{m-s}} \leq \sup_m \frac{\alpha_m}{\tilde{\alpha}_{m-t}} < \infty.$$

But, as we have seen, condition (iii) of Theorem 2 implies that

$$\lim_i \frac{\alpha_{m_i+1}}{\alpha_{m_i}} = \infty,$$

and from the definition of $\tilde{\alpha}$ it follows that $\tilde{\alpha}_{m_i+1-s} \leq \alpha_{m_i}$ for i large enough, and so

$$\lim_i \frac{\alpha_{m_i+1}}{\tilde{\alpha}_{m_i+1-s}} = \infty,$$

which is a contradiction.

REFERENCES

1. C. BESSAGA, Some remarks on Dragilev's theorem, *Studia Math.* **31** (1968), 307-318.
2. L. CRONE AND W. ROBINSON, Every nuclear Fréchet space with a regular basis has the quasi-equivalence property, *Studia Math.* **42** (1975), 203-207.
3. M. M. DRAGILEV, On regular bases in nuclear spaces, *Amer. Math. Soc. Trans.* **93** (1970), 61-82.
4. M. M. DRAGILEV, Köthe spaces differing in diametral dimensionality, *Sibersk. Mat. Zh. II* **3** (1970), 512-525. (Russian)
5. M. M. DRAGILEV, On special dimensions defined on some classes of Köthe spaces, *Mat. Sb.* **80** (1969), 213-228. (Russian)
6. A. PIETSCH, "Nukleare Lokalkonvexe Räume," Springer-Verlag, Berlin, 1965.
7. S. ROLEWICZ, On spaces of holomorphic functions, *Studia Math.* **21** (1961), 135-160.
8. V. P. ZAHARJUTA, On the isomorphism of Cartesian products of locally convex spaces, *Studia Math.* **46** (1973), 201-221.